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Seiberg duality as derived equivalence for some quiver gauge theories

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Abstract: We study Seiberg duality of quiver gauge theories associated to the complex cone over the second del Pezzo surface. Homomorphisms in the path algebra of the quivers in each of these cases satisfy relations which follow from a superpotential of the corresponding gauge theory as F-flatness conditions. We verify that Seiberg duality between each pair of these theories can be understood as a derived equivalence between the categories of modules of representation of the path algebras of the quivers. Starting from the projective modules of one quiver we construct tilting complexes whose endomorphism algebra yields the path algebra of the dual quiver. Finally, we present a general scheme for obtaining Seiberg dual quiver theories by constructing quivers whose path algebras are derived equivalent. We also discuss some combinatorial relations between this approach and some of the other approaches which has been used to study such dualities.

Keywords: D-branes, Brane Dynamics in Gauge Theories, Supersymmetry and Duality, Differential and Algebraic Geometry.

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1. Introduction

Seiberg duality relates two $\mathsf{N} = 1$ supersymmetric gauge theories with product gauge groups [1]–[4]. Two Seiberg dual theories have gauge groups with factors of different ranks N_c, but the same number of flavors N_f. A pair of theories which are Seiberg dual to each other belong to the same universality class; that is, they flow to the same conformal theory in the infra-red (IR). A physical way to realise this duality is by means of the suspended brane configuration [5, 6, 7]. In this scheme first a gauge theory is realised as the world volume theory of D-branes suspended between an NS5 and an NS5′ brane. Seiberg duality is then interpreted as a transporting the NS5 brane through the adjacent NS5′ brane, by invoking the so called s-rules.

An interesting class of theories where explicit constructions and tests of Seiberg duality have already been carried out successfully consists of quiver gauge theories. Some quiver gauge theories [8] can be obtained as world-volume theories of D-branes on an orbifold. Seiberg duality of a pair of such theories relates the corresponding D-branes, which in turn correlates this field-theoretic duality to the geometry of the moduli space of the gauge theory. Indeed, two Seiberg dual quiver gauge theories of D-branes turn out to have the same vacuum moduli space. In the special case in which the rank of the gauge group on the *dualising node*, the one selected to perform Seiberg duality, remains unchanged, Seiberg duality is amenable to a toric description. This happens if the rank of the gauge

group associated to this node, say i_0, and the corresponding flavors are related as $N_f(i_0) = 2N_c(i_0)$. The toric description derives from the interpretation of the vacuum moduli space of the associated gauge theories of D-branes as toric varieties. In such cases, where both theories of the dual couple have a toric moduli space, the actions of the algebraic torus $\mathbb{C}^\star$ on the toric varieties restrict the pair of gauge groups to be U(1) in the abelian and SU(N) in the non-abelian case, with the same N. This special case of Seiberg duality is known as *toric duality* [9]. Many examples of Seiberg dual pairs of D-brane gauge theories have been constructed using toric duality [10, 9, 11, 12, 13, 14]. The basic idea underlying toric duality is to construct a quiver gauge theory as the world-volume gauge theory of D-branes on the three-dimensional orbifold $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$ [12, 15]. Then by considering different partial resolutions of this orbifold one manufactures different gauge theories with the same IR limit, namely the theory at the orbifold. The complex cones over the zeroth Hirzebruch surface and the second del Pezzo surface arising from such partial resolutions each gives rise to a pair of toric dual theories [9, 11, 16]. The complex cone over the third del Pezzo surface, arising from another set of partial resolutions of $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$ gives rise to four toric dual theories instead [11, 10]. We shall summarise the rules for obtaining toric dual theories in the next section.

There is another approach to Seiberg duality, however [17]. The main idea behind this approach is to view a pair of Seiberg dual theories as the very same gauge theory, but constructed in terms of a different set of basic D-branes. The category of D-branes in $\mathsf{N} = 1$ supersymmetric quiver gauge theories, many of which are realised on an orbifold, is the bounded derived category of representations of the the path-algebra of the quiver. In this point of view it is possible to part company with the orbifolds and consider manifestations of Seiberg duality in this category. It turns out that Seiberg duality is a derived functor in this category — a pair of Seiberg dual theories are derived equivalent. This approach uses Rickard's theorem [18] on derived equivalence of the bounded derived category of representations of algebras. Let $D^b(\mathcal{A}\text{-mod})$ denote the bounded derived category of representations of an algebra $\mathcal{A}$ over the field of complex numbers, $\mathbb{C}$ and let T be a complex in $D^b(\mathcal{A}\text{-mod})$. The complex T is called a *tilting complex* if the algebra $\mathrm{Hom}\,(T, T)$ has finite global dimension, the direct summands of copies of T generate $D^b(\mathcal{A}\text{-mod})$ and its self-extension vanishes, $\mathrm{Ext}(T, T) = 0$. Then, if the endomorphism algebra $\mathrm{End}\,T$ of a tilting complex T in $D^b(\mathcal{A}\text{-mod})$ coincides with an algebra $\mathcal{B}$, the algebras $\mathcal{A}$ and $\mathcal{B}$ are derived equivalent. Applying this to the path algebras of quivers, if $\mathcal{A} = \mathbb{C}Q$, the path algebra of a quiver Q over $\mathbb{C}$ and $\mathcal{B} = \mathbb{C}Q'$, then these two path algebras are derived equivalent if a tilting complex T can be constructed in $D^b(\mathbb{C}Q\text{-mod})$ whose endomorphism algebra coincides with $D^b(\mathbb{C}Q'\text{-mod})$ and vice-versa. The quivers Q and Q' will be termed *dual* in the sequel and we shall use the shorthand $D(Q)$ for $D^b(\mathbb{C}Q\text{-mod})$. It has been conjectured [17] that two gauge theories corresponding to quivers whose path algebras are derived equivalent in this sense are Seiberg dual to each other. Let us stress at this point that this way of obtaining Seiberg dual theories does not depend on the toric description of the moduli space of the gauge theories. Thus, it is possible to deal with theories with $N_f \neq 2N_c$. Furthermore, in a physically interesting gauge theory one has, apart from the "quarks" spanning the path algebra of the quiver, relations which follow from a superpo-

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tential as F-flatness conditions. It has been explicitly checked in certain examples [19] that the dual fields in the gauge theory corresponding to the dual quiver constructed through derived equivalence satisfy relations that follow from the F-flatness conditions of the *dual superpotential*, that is the superpotential of the dual gauge theory. The dual superpotential is a deformation of the original superpotential one start with, by relevant operators, expressed in terms of dual fields, as we describe below. What is more, the dual superpotential has but cubic interactions, thereby guaranteeing $\mathsf{N} = 1$ supersymmetry of the dual gauge theory. These examples include the quivers arising from the complex cone over the zeroth Hirzebruch surface studied earlier [9, 11].

In this note we verify this conjecture for the quivers associated to the second del Pezzo surface. There are only two toric phases for this case [11]. Hence there are only two toric dual theories. However, once we relax the restriction on the theory to be toric, i.e. allow for $N_f \neq 2N_c$ on the dualising node, we show by explicit construction that there are other theories, which are Seiberg dual to each other. For all these cases we construct tilting complexes starting from one and find out the quivers whose path algebra coincides with the endomorphism algebra of the tilting complex, that is, the dual quivers. We verify that the associated gauge theories are Seiberg dual to each other and are compatible with the known rules laid down earlier [20]. These non-toric theories are of import in studying renormalisation group flow and the resulting cascade of Seiberg dual theories [21, 20, 22]. The toric as well as the non-toric phases arise along such a flow in general.

The lay out of this note is as follows. In the next section we set up notations and recall the rules for performing Seiberg duality operations on a quiver in general. In section 3 we discuss Seiberg duality of the quivers associated with the second del Pezzo surface. We explicitly construct tilting complexes and derive the endomorphism algebras by taking each of the five nodes of the starting quiver in turn. In section 4 we present a general method for obtaining Seiberg dual theories based on the examples of section 3. We summarise the results in section 5.

2. Seiberg duality of quiver gauge theories

In this note we restrict our discussions to $\mathsf{N} = 1$ quiver gauge theories, which have gauge groups of the form $G = \prod_i \mathrm{SU}(N_c(i))$ and flavors $N_f(i)$ associated to each node i of the quiver. A quiver gauge theory contains chiral multiplets, which we shall refer to as *quarks* at times. While each factor in the gauge group G corresponds to a node in the quiver, each arrow corresponds to a chiral field transforming in the bi-fundamental representation of the two factors on the two nodes it connects. We use the convention that a chiral field χ transforming in the fundamental representation of a factor $\mathrm{SU}(N_c(i))$ and in the anti-fundamental of the (possibly same) factor $\mathrm{SU}(N_c(j))$ in G will be represented by an arrow from the j-th node to the i-th node of the quiver, and will be denoted χ_{ij}, as $\textcircled{j} \xrightarrow{\chi_{ij}} \textcircled{i}$. For the morphisms between projectives, we use the convention [19] that morphisms are along the arrows of the quiver.

Seiberg duality provides a prescription to relate the chiral operators of two theories. This can thus be used to map the deformations of one theory to that of the other in such

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a way that the low-energy (IR) properties of the pair remain unaltered. Let us briefly recall how this works [10]. Starting with a quiver gauge theory one chooses a factor in the gauge group G, say $\mathrm{SU}(N_c(i_0))$, corresponding to a node i_0 in the quiver. The chiral fields transforming in the fundamental corresponding to this factor and represented by incoming arrows on the node i_0 are singled out. The theory is then *deformed* by turning off the superpotential couplings of these fields. The gauge couplings of all the factors $\mathrm{SU}(N_c(i))$, $i \neq i_0$, in G under which these fields are charged are sent to infinity as well, i.e. $1/g_i^2 \longrightarrow 0$. Seiberg duality then predicts an equivalent description of this deformed theory in terms of certain dual fields transforming under a gauge group of possibly different rank, $\mathrm{SU}(N_c'(i_0))$, where $N_c'(i_0) = N_f(i_0) - N_c(i_0)$. The deformed theory is written in terms of these dual variables, called *dual quarks*, and certain composites of the original fields, called *mesons*. The couplings are then restored to generic values, thus obtaining a dual description of the original theory, with a possibly different gauge group. Let us start by recalling the rules for obtaining Seiberg dual theories starting from a given quiver gauge theory [20]. As mentioned above, in obtaining a Seiberg dual theory, one singles out a node i_0 of the quiver as the dualising node. Seiberg duality alters the rank of the gauge group associated to this node, as well as the quarks in the theory. First, one makes certain combinatorial operations on the quiver ignoring the superpotential. After obtaining the dual quiver, the consistency of the dual theory is checked by verifying that the fields in this theory satisfy F-flatness conditions of a cubic superpotential. Apart from changing the rank of the gauge group $\mathrm{SU}(N_c(i_0))$, these operations modifies the adjacency matrix C of the quiver, whose entries C_{ij} denote the number of arrows from the i-th to the j-th node. Let us point out that to start with we can write the adjacency matrix as one with all entries positive, counting only the arrows emanating from the nodes. Then the quiver of the Seiberg dual theory can be obtained by applying the following rules.

Let i_0 denote the dualising node. Let the set of nodes from which the incoming arrows on i_0 originate be denoted I_{in} and the set of nodes on which outgoing arrows from i_0 terminate be denoted I_{out}. The Seiberg dual quiver is obtained by

- changing the rank $N_c(i_0)$ of the gauge group factor on the node i_0 to $N_c'(i_0) = N_f(i_0) - N_c(i_0)$, with $N_f(i_0) = \sum_{j \in I_{\mathrm{in}}} C_{ji_0} N_c(j)$. The new node is interpreted as an anti-brane of the original one, and

- deriving the adjacency matrix $\widetilde{C}$ of the dual quiver from that of the original quiver as,

$$\tilde{C}_{ij} = \begin{cases} C_{ji}\,, & \text{if either } i_0 = i \quad \text{or } i_0 = j\,, \\ C_{ij} - C_{ji_0}C_{i_0 i}\,, & \text{if } i \in I_{\mathrm{out}} \;\&\; j \in I_{\mathrm{in}}\,, \\ C_{ij}\,, & \text{otherwise}\,. \end{cases} \tag{2.1}$$

A negative entry in the dual adjacency matrix is drawn as a reversed arrow on the dual quiver. We shall see that the entries of $\widetilde{C}_{ij}$ count the morphisms between complexes in $\mathrm{End}\,T$ in our examples.

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3. Quiver theories from the second del Pezzo surface

In this section we find out examples of Seiberg dual theories by explicitly constructing tilting complexes and working out their endomorphism algebras. In particular, we consider the quiver theories associated to the complex cone over the second del Pezzo surface, which arise in a certain partial resolution of $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$. Another partial resolution of $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$ yields the complex cone over the zeroth Hirzebruch surface, for which toric duality predicts two toric phases and hence two toric quiver theories [12]. That these two theories are derived equivalent has also been established [19]. The quivers in this example have a simple pattern. There are four nodes in the quivers and one of the quivers have the feature that two arrows originate from each node to go to the adjacent node while two arrows from the other adjacent node terminate on it. Thus, taking any of the four nodes as the dualising node leads to the same dual quiver up to permutations of nodes. This does not hold in the present case. The quivers associated to the complex cone over the second del Pezzo surface have five nodes each. Again, one of them is rather symmetric in the sense that two arrows emanate from each node and two arrows terminate on each. But they do not connect only the adjacent nodes, nor the pair of emanating or terminating arrows connect to the same nodes. This is the quiver, shown in figure 1, that we begin with. The other quivers, as we find, do not have this feature. We then take each node of the quiver in figure 1 in turn as the dualising node and write down the adjacency matrix of the dual quiver using the rules (2.1). We then present a tilting complex for each case whose endomorphism algebra coincides with the path algebra of the dual quiver. The homomorphisms between the direct summands of the tilting complex in each case have to satisfy certain relations, namely the F-flatness conditions from the corresponding superpotential. We write down the dual superpotential in each case and verify that the homomorphisms we find do satisfy the F-flatness conditions of the dual superpotential. However, for the dual theory obtained upon dualising on the first node, we find that it is easier to find the dual superpotential by imposing the relations between the homomorphisms. Let us start by writing down the quiver data and the gauge theory data corresponding to the quiver Q in figure 1, which we sometimes refer to as *the original quiver*. The adjacency matrix is

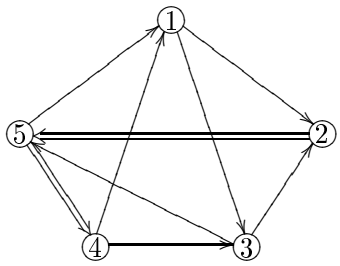

Figure 1: Quiver Q corresponding to the second del Pezzo surface.

$$C = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \end{pmatrix}. \tag{3.1}$$

The gauge group is taken to be $\prod_{i=1}^{5} \mathrm{SU}(n_i)$, that is $N_c(i) = n_i$ for $i = 1, \cdots, 5$, fixing the representation of the quiver. The requirement that the gauge theory is non-anomalous

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constraints the numbers n_i. Cancellation of anomaly requires

$$\sum_j (C - C^{\mathrm{T}})_{ij} \cdot N_c(j) = 0\,. \tag{3.2}$$

Using the adjacency matrix (3.1) this yields two relations among the five n's, namely,

$$n_1 + n_4 = n_2 + n_5\,, \qquad n_2 + n_3 = n_4 + n_5\,. \tag{3.3}$$

The superpotential of the $\mathsf{N} = 1$ gauge theory corresponding to the original quiver can be obtained as the superpotential of a partially resolved theory from the superpotential of the $\mathsf{N} = 1$ gauge theory on $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$. The superpotential is [12]

$$\mathcal{W} = y_{52}x_{23}x_{34}x_{45} + x_{14}y_{45}x_{52}x_{21} + x_{53}x_{31}x_{15} - y_{45}x_{53}x_{34} - x_{52}x_{23}x_{31}x_{14}x_{45} - x_{15}y_{52}x_{21}\,. \tag{3.4}$$

Each term in the superpotential is a trace over the combinations of chiral fields. We shall not mention the trace explicitly in this note. In (3.4) each x_{ij} or y_{ij} denote a bosonic component of the chiral fields of the $\mathsf{N} = 1$ theory. The subscripts indicate the nodes they connect, as mentioned in the last section.

We now proceed to Seiberg dualise this quiver node by node. Let us note at the outset that since there are five nodes in the quiver we expect at most five dual theories, including the one corresponding to the original quiver. Now, as we Seiberg dualise the five nodes, we obtain five dual theories, apart from the original one. We then expect one of these five new theories to coincide with the original one. Indeed, dualising on the first node we obtain a quiver which gives back the original quiver after re-labelling the nodes. Let us start with this case.

3.1 Dualising on the first node

We begin by taking the first node as the dualising node, that is, $i_0 = 1$. Then we have $I_{\mathrm{in}} = \{5, 4\}$ and $I_{\mathrm{out}} = \{2, 3\}$. Now applying the rules (2.1) on the node 1 we can find out the adjacency matrix of the dual quiver and its representation. Obviously, only the rank of the gauge group associated to this node may change. The adjacency matrix of the new quiver turns out to be

$$\widetilde{C} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}. \tag{3.5}$$

The corresponding quiver is shown in figure 2*b*. Let us recall that a negative value of C_{ij} is represented by an arrow from the j-th to the i-th node. Our goal is to re-derive this quiver from the endomorphism algebra of a tilting complex. For this let us start with the projectives P_i, $i = 1, \cdots, 5$, corresponding to the five nodes and consider the complex $T = \oplus_{i=1}^{5} T_i$ in the derived category $D(Q)$ of the original quiver Q, with direct summands

$$T_1: \quad 0 \longrightarrow \underline{P_4 \oplus P_5} \xrightarrow{\binom{x_{14}}{-x_{15}}} P_1 \longrightarrow 0$$

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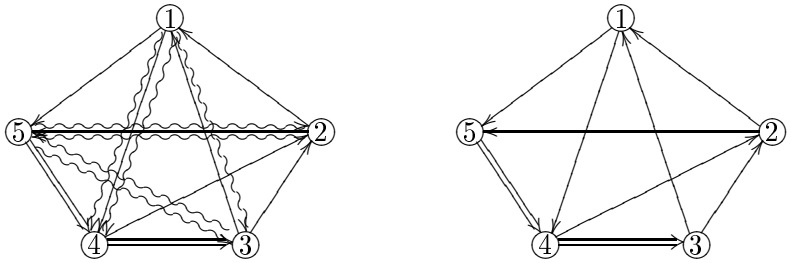

(a) Dual quiver with massive fields. (b) Dual quiver.

Figure 2: Quivers from Seiberg dualising on node 1.

	T_1	T_2	T_3	T_4	T_5
T_1	$(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),1)$		$X_{31}{:=}(\left(\begin{smallmatrix}x_{34}\\0\end{smallmatrix}\right),0)$	$X_{41}{:=}(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),0)$ $Y_{41}{:=}(\left(\begin{smallmatrix}0\\x_{45}\end{smallmatrix}\right),0)$ $Z_{41}{:=}(\left(\begin{smallmatrix}0\\y_{45}\end{smallmatrix}\right),0)$	$X_{51}{:=}(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),0)$
T_2	$X_{12}{:=}(\left(\begin{smallmatrix}y_{45}x_{52}\\y_{52}\end{smallmatrix}\right),0)$	$(1,0)$			$X_{52}{:=}(x_{52},0)$ $Y_{52}{:=}(y_{52},0)^\star$
T_3	$X_{13}{:=}(\left(\begin{smallmatrix}x_{45}x_{52}x_{23}\\x_{53}\end{smallmatrix}\right),0)$	$X_{23}{:=}(x_{23},0)$	$(1,0)$		$X_{53}{:=}(x_{53},0)^\star$
T_4		$X_{24}{:=}(x_{21}x_{14},0)$	$X_{34}{:=}(x_{34},0)$ $Y_{34}{:=}(x_{31}x_{14},0)$	$(1,0)$	
T_5		$X_{25}{:=}(x_{21}x_{15},0)^\star$	$X_{35}{:=}(x_{31}x_{15},0)^\star$	$X_{45}{:=}(x_{45},0)$ $Y_{45}{:=}(y_{45},0)$	$(1,0)$

Table 1: End T for dualising on node 1. Massive fields marked $\star$.

$$\begin{aligned} T_2: &\quad 0 \longrightarrow \underline{P_2} \longrightarrow 0 \\ T_3: &\quad 0 \longrightarrow \underline{P_3} \longrightarrow 0 \\ T_4: &\quad 0 \longrightarrow \underline{P_4} \longrightarrow 0 \\ T_5: &\quad 0 \longrightarrow \underline{P_5} \longrightarrow 0\,. \end{aligned} \tag{3.6}$$

The zeroth position in the complexes are underlined. It can be checked that T is a tilting complex, $\mathrm{Ext}(T_i, T_j) = 0$ for all $i, j = 1, \ldots, 5$. We then evaluate the non-zero homomorphisms between each pair of T_i in the derived category $D(Q)$, thus finding the endomorphism algebra of T. The result is summarised in table 1. The entries which are absent from the table are zero maps in the corresponding homotopy category. The homomorphisms listed in the table are the dual fields in the gauge theory — dual quarks and dual mesons — and the dual superpotential has to be expressed in terms of these fields. Drawing arrows corresponding to the entries in the table we obtain the quiver figure 2a. It can be checked from the table that $X_{31} = X_{34}X_{41}$, $Y_{41} = X_{45}X_{51}$ and $Z_{41} = Y_{45}X_{51}$. Thus these three homomorphisms are determined by the others and the corresponding fields are absent from the superpotential. Moreover, from the superpotential $\mathcal{W}$ in (3.4) we note that $x_{31}x_{15}x_{53} = X_{35}X_{53}$ and $y_{52}x_{21}x_{15} = Y_{52}X_{25}$, thus giving mass to the four dual fields. In the table they are marked with an asterisk ($\star$). These seven fields are to be dropped out from the quiver and shown as wiggly lines in figure 2a. Leaving out these fields the resulting quiver is as shown in figure 2b, the dual quiver that we obtained using the rules (2.1) above.

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Let us briefly digress to discuss the combinatorics of the above calculations with an example. From the rules (2.1), we have $\widetilde{C}_{ij} = C_{ji}$ when $i = 1$ or $j = 1$. The effect of this is to invert the arrows on the node 1. In writing the tilting complex T this has been achieved by shifting the projective P_1 in the summand T_1 by one degree. Indeed, a one-degree shift is the operation in the derived category that takes a brane to its anti-brane [23]. Furthermore, we have, for example, $\widetilde{C}_{25} = C_{25} - C_{51}C_{12} = 2 - 1$ from (2.1), while $\widetilde{C}_{52} = C_{52} = 0$. This shows that one of the two arrows from node 2 to node 5 is "killed" by an arrow which is a composite of arrows from node 5 to node 1 and from 1 to 2. The "killer" and the "victim" are represented by the wiggly lines in figure 2*a*. Both of them are eliminated in the dual quiver (none of them may be around at large, after all!). From table 1 it can be checked that the fields corresponding to the homomorphisms Y_{52} and X_{25}, the latter being the same composite as stated above, become massive together. Similar combinatorics apply to the evaluation of $\mathrm{End}\, T$ in all the cases in this note.

We still need to check whether the morphisms listed in the table satisfy the relations derived from F-flatness conditions of the dual superpotential. For this we first need to write the dual potential by expressing $\mathcal{W}$ in terms of the dual fields from the table and then adding to it the "mesonic" part [10, 12]. Finally we integrate out the fields which gain superpotential masses to derive the dual superpotential. Thus, let us first rewrite the superpotential $\mathcal{W}$ in terms of the dual fields. To this end, let us first note that the dual quarks are X_{12}, X_{13}, X_{41} and X_{51}, that is the inverted arrows on the node 1. The mesons are the composites, X_{24}, X_{25}, Y_{34} and X_{35}. The superpotential $\mathcal{W}$ expressed in terms of the dual fields from table 1 assumes the form

$$\mathcal{W}' = Y_{52}X_{23}X_{34}X_{45} + Y_{45}X_{52}X_{24} + X_{53}X_{35} - Y_{45}X_{53}X_{34} - X_{52}X_{23}Y_{34}X_{45} - Y_{52}X_{25}\,. \quad (3.7)$$

Let us point out that none of the fields in the above expression correspond to an arrow which either originates or terminates on the dualising node 1. That is, the loops corresponding to the terms do not go through the first node. It is clear from the expression for $\mathcal{W}'$ that the fields X_{53}, X_{35}, Y_{52} and X_{25} are massive, as indicated in table 1. Now, we write the new part of the superpotential involving the mesons. The particular combinations appearing in the superpotential and the signs of the terms in this part can be fixed by noting that

- each field occurs twice and only twice in the superpotential,
- two terms involving any given field appear with opposite signs,
- any pair of terms may have only a single field in common.

Using these rules, we get the "mesonic" part of the superpotential

$$\mathcal{W}_m = X_{13}Y_{34}X_{41} + X_{12}X_{25}X_{51} - X_{12}X_{24}X_{41} - Y_{13}X_{35}X_{51}\,, \quad (3.8)$$

in which each term corresponds to a loop going through the *new* first node. We then integrate out the massive fields from the new superpotential $\mathcal{W}' + \mathcal{W}_m$ by using their equations of motion, for example,

$$X_{25} = X_{23}X_{34}X_{45}\,, \quad (3.9)$$

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obtained from $\partial\widetilde{\mathcal{W}}/\partial Y_{52} = 0$ etc. Finally we obtain the dual superpotential

$$\begin{aligned}\widetilde{\mathcal{W}} = {} & X_{12}X_{23}X_{34}X_{45}X_{51} + Y_{45}X_{52}X_{24} + X_{13}Y_{34}X_{41} - \\ & -Y_{45}X_{51}X_{13}X_{34} - X_{51}X_{23}Y_{34}X_{45} - X_{12}X_{24}X_{41}\,. \end{aligned} \tag{3.10}$$

It can now be verified, by using the expressions in table 1 and the F-flatness conditions of the original superpotential, that the dual fields satisfy the eleven F-term equations coming from $\widetilde{\mathcal{W}}$. Let us discuss the two non-trivial cases that need be treated specially.

The expression $\partial\widetilde{\mathcal{W}}/\partial X_{13} = Y_{34}X_{41} - X_{34}Y_{45}X_{51}$, in terms of the original fields is

$$\begin{pmatrix} x_{31}x_{14} \\ -x_{34}y_{45} \end{pmatrix} = x_{31}\begin{pmatrix} x_{14} \\ -y_{15} \end{pmatrix}, \tag{3.11}$$

where we used the F-term equation, $\partial\mathcal{W}/\partial x_{53} = x_{31}x_{15} - x_{34}y_{45} = 0$. While not zero as it is, this expression is homotopic to zero with homotopy $(\ldots, 0, x_{31}, 0, \ldots)$, as can be seen from the following diagram

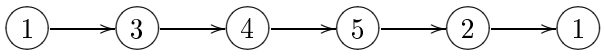
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{P_4 \oplus P_5} & \xrightarrow{\left(\begin{smallmatrix} x_{14} \\ -x_{15} \end{smallmatrix}\right)} & P_1 & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix} x_{31}x_{14} \\ -x_{31}x_{15} \end{smallmatrix}\right)} & \swarrow{\scriptstyle x_{31}} & & & \\ 0 & \longrightarrow & \underline{P_3} & \longrightarrow & 0 & & \end{array} \tag{3.12}$$

Thus, $\partial\widetilde{\mathcal{W}}/\partial X_{13} = 0$ in the homotopy category of the complexes $\{T_i, i = 1, \ldots, 5\}$ and hence in the resulting derived category.

The expression $\partial\widetilde{\mathcal{W}}/\partial X_{12} = X_{23}X_{34}X_{45}X_{51} - X_{24}X_{41}$ can similarly be shown to be zero in the derived category, using the relation $\partial\mathcal{W}/\partial x_{52} = x_{23}x_{34}x_{45} - x_{21}x_{15} = 0$ and the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{P_4 \oplus P_5} & \xrightarrow{\left(\begin{smallmatrix} x_{14} \\ -x_{15} \end{smallmatrix}\right)} & P_1 & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix} x_{21}x_{14} \\ -x_{21}x_{15} \end{smallmatrix}\right)} & \swarrow{\scriptstyle x_{21}} & & & \\ 0 & \longrightarrow & \underline{P_2} & \longrightarrow & 0 & & \end{array} \tag{3.13}$$

As mentioned earlier, the quiver figure 2*b* can be mapped to the original quiver figure 1 by the following re-labelling of the nodes:



3.2 Dualising on the second node

Let us now perform Seiberg duality on the same quiver Q in figure 1 taking the node 2 as the dualising node. Again, applying the rules (2.1) to calculate the adjacency matrix of the dual quiver we obtain

$$\widetilde{C} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 2 & 0 \end{pmatrix}. \tag{3.14}$$

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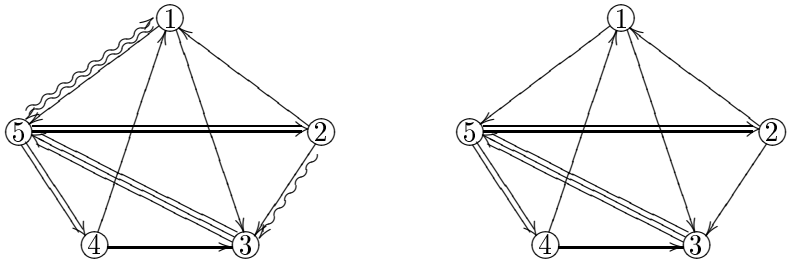

(*a*) Dual quiver with massive fields. (*b*) Dual quiver.

Figure 3: Quivers from Seiberg dualising on node 2.

	T_1	T_2	T_3	T_4	T_5
T_1	$(1,0)$		$X_{31}{:=}(x_{31},0)$		$X_{51}{:=}(x_{52}x_{21},0)$ $Y_{51}{:=}(y_{52}x_{21},0)^\star$
T_2	$X_{12}{:=}\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),0\right)$	$\left(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),1\right)$	$X_{32}{:=}\left(\left(\begin{smallmatrix}x_{31}\\0\end{smallmatrix}\right),0\right)$ $Y_{32}{:=}\left(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),0\right)$		$X_{52}{:=}\left(\left(\begin{smallmatrix}0\\x_{53}\end{smallmatrix}\right),0\right)$
T_3			$(1,0)$		$X_{53}{:=}(x_{53},0)$ $Y_{53}{:=}(x_{52}x_{23},0)$ $Z_{53}{:=}(y_{52}x_{23},0)$
T_4	$X_{14}{:=}(x_{14},0)$		$X_{34}{:=}(x_{34},0)$	$(1,0)$	
T_5	$X_{15}{:=}(x_{15},0)^\star$	$X_{25}{:=}\left(\left(\begin{smallmatrix}x_{15}\\x_{34}x_{45}\end{smallmatrix}\right),0\right)$ $Y_{25}{:=}\left(\left(\begin{smallmatrix}x_{14}y_{45}\\x_{31}x_{14}x_{45}\end{smallmatrix}\right),0\right)$		$X_{45}{:=}(x_{45},0)$ $Y_{45}{:=}(y_{45},0)$	$(1,0)$

Table 2: End T for dualising on node 2. Massive fields marked $\star$.

The resulting quiver with the massive fields is shown in figure 3*a*, while the dual quiver is shown in figure 3*b*. The details of the calculation is similar to the previous case and we shall not discuss them here. We reproduce the path algebra of the quiver figure 3*b* as the endomorphism algebra of the tilting complex $T = \oplus_{i=1}^{5} T_i$, with the direct summands

$$\begin{aligned}
T_1 :&\ 0 \longrightarrow \underline{P_1} \longrightarrow 0\\
T_2 :&\ 0 \longrightarrow \underline{P_1 \oplus P_3} \xrightarrow{\left(\begin{smallmatrix}x_{21}\\-x_{23}\end{smallmatrix}\right)} P_2 \longrightarrow 0\\
T_3 :&\ 0 \longrightarrow \underline{P_3} \longrightarrow 0\\
T_4 :&\ 0 \longrightarrow \underline{P_4} \longrightarrow 0\\
T_5 :&\ 0 \longrightarrow \underline{P_5} \longrightarrow 0\,.
\end{aligned} \tag{3.15}$$

The morphisms in End$T$ are shown in table 2. From the table, we see that $X_{32} = X_{31}X_{12}$ and $X_{52} = X_{53}Y_{32}$, and are therefore dropped from the quiver as well as from the dual superpotential. Moreover, from the superpotential $\mathcal{W}$, the term $x_{15}y_{52}x_{21} = Y_{51}X_{15}$. This makes the two dual quarks massive. These are marked with an asterisk in the above table.

The calculation of the dual superpotential proceeds as in the previous case. In this case, the dual quarks are X_{12}, Y_{32}, X_{25} and Y_{25}, while the mesons are X_{51} Y_{51}, Y_{53} and Z_{53}. As in the previous case, we derive the dual superpotential for the present example

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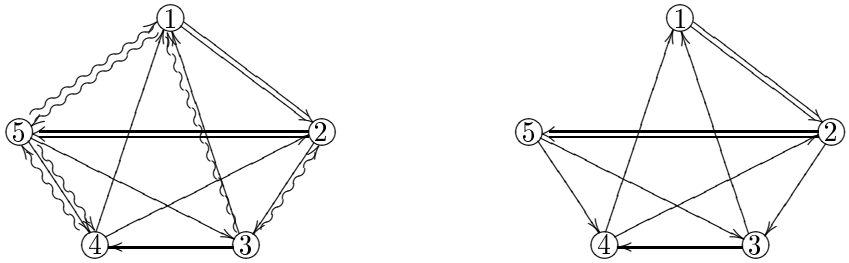

(*a*) Dual quiver with massive fields. (*b*) Dual quiver with massive fields.

Figure 4: Quivers from Seiberg dualising on node 3.

to be

$$\begin{aligned}\widetilde{\mathcal{W}} = {} & Z_{53}X_{34}X_{45} + X_{14}Y_{45}X_{51} + X_{25}X_{53}X_{31}X_{12} + Y_{25}Y_{53}Y_{32} - \\ & - Y_{45}X_{53}X_{34} - Y_{53}X_{31}X_{14}X_{45} - Y_{25}X_{51}X_{12} - X_{25}Z_{53}Y_{32}\,. \end{aligned} \tag{3.16}$$

The relations ensuing from the F-flatness conditions of the superpotential $\widetilde{\mathcal{W}}$ are satisfied by the morphisms in table 2 upon using the F-flatness conditions of the original superpotential $\mathcal{W}$. Two of the cases are non-trivial. These are

$$\frac{\partial\widetilde{\mathcal{W}}}{\partial X_{25}} = \begin{pmatrix} y_{52}x_{21} \\ -y_{52}x_{23} \end{pmatrix} \qquad \text{and} \qquad \frac{\partial\widetilde{\mathcal{W}}}{\partial Y_{25}} = \begin{pmatrix} x_{52}x_{21} \\ -x_{52}x_{23} \end{pmatrix}, \tag{3.17}$$

which can be seen to be zero in the derived category from the following diagrams respectively

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{P_1 \oplus P_3} & \xrightarrow{\binom{x_{21}}{-x_{23}}} & P_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \binom{y_{52}x_{21}}{-y_{52}x_{23}}} & \swarrow{\scriptstyle y_{52}} & & & \\ 0 & \longrightarrow & \underline{P_5} & \longrightarrow & 0 & & \end{array} \qquad \text{and} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \underline{P_1 \oplus P_3} & \xrightarrow{\binom{x_{21}}{-x_{23}}} & P_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \binom{x_{52}x_{21}}{-x_{52}x_{23}}} & \swarrow{\scriptstyle x_{52}} & & & \\ 0 & \longrightarrow & \underline{P_5} & \longrightarrow & 0 & & \end{array} \tag{3.18}$$

3.3 Dualising on the third node

We repeat the exercise of performing Seiberg duality with now the third node as the dualising node. The calculations are similar to the previous cases and we only present the results. The adjacency matrix of the dual quiver obtained by applying the rules (2.1) is

$$\widetilde{C} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 2 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}. \tag{3.19}$$

The dual quivers, with and without the massive fields, are shown in figure 4 as before. The tilting complex T in the derived category $D(Q)$ of the original quiver is taken as $T = \oplus_{i=1}^{5} T_i$ with the direct summands

$$T_1 : \quad 0 \longrightarrow \underline{P_1} \longrightarrow 0$$

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	T_1	T_2	T_3	T_4	T_5
T_1	$(1,0)$	$X_{21}{:=}(x_{21},0)$ $Y_{21}{:=}(x_{23}x_{31},0)$			$X_{51}{:=}(x_{53}x_{31},0)^\star$
T_2		$(1,0)$	$X_{32}{:=}\left(\left(\begin{smallmatrix} x_{14}x_{45}x_{52} \\ x_{45}y_{52} \end{smallmatrix}\right),0\right)$		$X_{52}{:=}(x_{52},0)$ $Y_{52}{:=}(y_{52},0)$
T_3	$X_{13}{:=}\left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right),0\right)$ $Y_{13}{:=}\left(\left(\begin{smallmatrix} 0 \\ x_{14} \end{smallmatrix}\right),0\right)$	$X_{23}{:=}\left(\left(\begin{smallmatrix} x_{21} \\ 0 \end{smallmatrix}\right),0\right)$	$\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right),1\right)$	$X_{43}{:=}\left(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right),0\right)$	
T_4	$X_{14}{:=}(x_{14},0)$	$X_{24}{:=}(x_{23}x_{34},0)$		$(1,0)$	$X_{54}{:=}(x_{53}x_{34},0)^\star$
T_5	$X_{15}{:=}(x_{15},0)^\star$		$X_{35}{:=}\left(\left(\begin{smallmatrix} x_{15} \\ y_{45} \end{smallmatrix}\right),0\right)$	$X_{45}{:=}(x_{45},0)$ $Y_{45}{:=}(y_{45},0)^\star$	$(1,0)$

Table 3: End T for dualising on node 3. Massive fields marked $\star$.

$$\begin{aligned}
T_2:&\quad 0\longrightarrow \underline{P_2}\longrightarrow 0\\
T_3:&\quad 0\longrightarrow \underline{P_1\oplus P_4}\xrightarrow{\left(\begin{smallmatrix} x_{31} \\ -x_{34} \end{smallmatrix}\right)} P_3\longrightarrow 0\\
T_4:&\quad 0\longrightarrow \underline{P_4}\longrightarrow 0\\
T_5:&\quad 0\longrightarrow \underline{P_5}\longrightarrow 0\,.
\end{aligned} \tag{3.20}$$

The morphisms in the endomorphism algebra $\mathrm{End}\,T$ are tabulated in table 3. Here again we find that the two fields $X_{23} = X_{21}X_{13}$ and $Y_{13} = X_{14}X_{43}$ are determined by others and hence dropped. Moreover, from the terms $x_{53}x_{31}x_{15} = X_{51}X_{15}$ and $y_{45}x_{53}x_{34} = Y_{45}X_{54}$ in the superpotential $\mathcal{W}$, we see that four dual quarks become massive.

The dual superpotential is evaluated as before. In this case the dual quarks are X_{13}, X_{43}, X_{32} and X_{35}, while the mesons are X_{51}, X_{24}, Y_{21} and X_{54}. In the present case the dual superpotential takes the form

$$\begin{aligned}
\widetilde{\mathcal{W}} = {} & Y_{52}X_{24}X_{45} + X_{14}X_{43}X_{35}X_{52}X_{21} + X_{32}Y_{21}X_{13} - \\
& - X_{52}Y_{21}X_{14}X_{45} - X_{35}Y_{52}X_{21}X_{13} - X_{32}X_{24}X_{43}\,.
\end{aligned} \tag{3.21}$$

The corresponding relations from the F-flatness conditions of $\widetilde{\mathcal{W}}$ are satisfied by the morphisms in table 3 upon using the F-flatness conditions ensuing from $\mathcal{W}$. Again there are two non-trivial relations, namely,

$$\frac{\partial\widetilde{\mathcal{W}}}{\partial X_{35}} = \begin{pmatrix} -x_{53}x_{31} \\ x_{53}x_{34} \end{pmatrix} \qquad \text{and} \qquad \frac{\partial\widetilde{\mathcal{W}}}{\partial X_{21}} = \begin{pmatrix} x_{23}x_{31} \\ -x_{23}x_{34} \end{pmatrix}, \tag{3.22}$$

which can be shown to be zero in the derived category by considering the following diagrams respectively.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{P_1\oplus P_4} & \xrightarrow{\left(\begin{smallmatrix} x_{31} \\ -x_{34} \end{smallmatrix}\right)} & P_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \left(\begin{smallmatrix} -x_{53}x_{31} \\ x_{53}x_{34} \end{smallmatrix}\right)} & \swarrow{\scriptstyle -x_{53}} & & & \\
0 & \longrightarrow & \underline{P_5} & \longrightarrow & 0 & &
\end{array} \qquad \text{and} \qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{P_1\oplus P_4} & \xrightarrow{\left(\begin{smallmatrix} x_{31} \\ -x_{34} \end{smallmatrix}\right)} & P_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \left(\begin{smallmatrix} x_{23}x_{31} \\ -x_{23}x_{34} \end{smallmatrix}\right)} & \swarrow{\scriptstyle x_{23}} & & & \\
0 & \longrightarrow & \underline{P_2} & \longrightarrow & 0 & &
\end{array} \tag{3.23}$$

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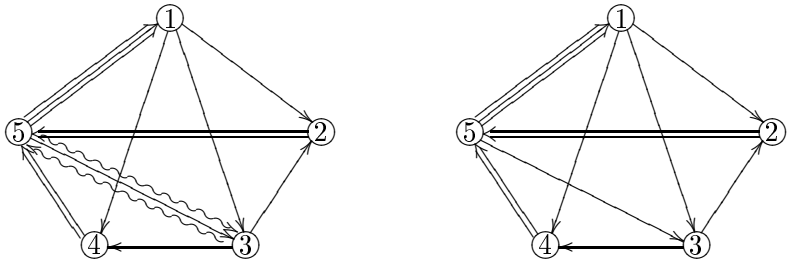

(a) Dual quiver with massive fields. (b) Dual quiver.

Figure 5: Quivers from Seiberg dualising on node 4.

3.4 Dualising on the fourth node

Next, by choosing the fourth node for dualisation we obtain the adjacency matrix of the dual quiver

$$\widetilde{C} = \begin{pmatrix} 0 & 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.24}$$

by applying the rules (2.1). This furnishes the dual quiver shown in figure 5b. The quiver with massive fields is shown in figure 5a. The gauge theory corresponding to the dual quiver figure 5b is a toric one found earlier through toric duality [12]. The path algebra of the quiver figure 5b coincides with the endomorphism algebra of the tilting complex $T = \oplus_{i=1}^{5} T_i$ in $D(Q)$, with direct summands

$$\begin{aligned} T_1 &: \ 0 \longrightarrow \underline{P_1} \longrightarrow 0 \\ T_2 &: \ 0 \longrightarrow \underline{P_2} \longrightarrow 0 \\ T_3 &: \ 0 \longrightarrow \underline{P_3} \longrightarrow 0 \\ T_4 &: \ 0 \longrightarrow \underline{P_5 \oplus P_5} \xrightarrow{\binom{x_{45}}{-y_{45}}} P_4 \longrightarrow 0 \\ T_5 &: \ 0 \longrightarrow \underline{P_5} \longrightarrow 0 \,. \end{aligned} \tag{3.25}$$

The morphisms in $\mathrm{End}\, T$ are shown in table 4. We have the fields $X_{14} = X_{15}Y_{54}$ and $Y_{14} = X_{15}X_{54}$ determined in terms of others and hence dropped. Moreover, in the original superpotential $\mathcal{W}$, the term $y_{45}x_{53}x_{34} = X_{53}Y_{35}$, which makes the two dual quarks massive. In this case the surviving dual quarks are X_{54}, Y_{54}, X_{41} and X_{43}, while the mesons are Y_{15}, Z_{15}, X_{35} and Y_{35}. The dual superpotential is now given by

$$\begin{aligned} \widetilde{\mathcal{W}} = {} & X_{43}X_{35}Y_{54} + X_{15}Y_{52}X_{21} + X_{41}Z_{15}X_{54} - X_{35}Y_{52}X_{23} - \\ & - X_{41}Y_{15}Y_{54} - X_{15}X_{54}X_{43}X_{31} - Z_{15}X_{52}X_{21} - X_{31}Y_{15}X_{52}X_{23} \,. \end{aligned} \tag{3.26}$$

Again, the maps from table 4 satisfy the corresponding F-flatness conditions. The non-trivial ones are

$$\frac{\partial \widetilde{\mathcal{W}}}{\partial X_{41}} = \begin{pmatrix} x_{14}x_{45} \\ -x_{14}y_{45} \end{pmatrix} \qquad \text{and} \qquad \frac{\partial \widetilde{\mathcal{W}}}{\partial X_{43}} = \begin{pmatrix} x_{34}x_{45} \\ -x_{34}y_{45} \end{pmatrix} \tag{3.27}$$

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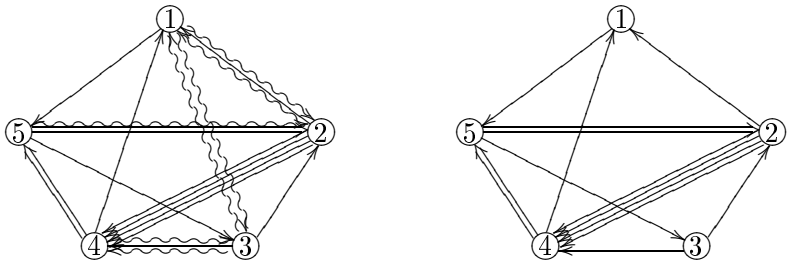
	T_1	T_2	T_3	T_4	T_5
T_1	$\binom{1}{0}$	$X_{21}{:=}(x_{21},0)$	$X_{31}{:=}(x_{31},0)$	$X_{41}{:=}\left(\binom{x_{52}x_{23}x_{31}}{x_{52}x_{21}},0\right)$	
T_2		$(1,0)$			$X_{52}{:=}(x_{52},0)$ $Y_{52}{:=}(y_{52},0)$
T_3		$X_{23}{:=}(x_{23},0)$	$(1,0)$	$X_{43}{:=}\left(\binom{y_{52}x_{23}}{x_{53}},0\right)$	$X_{53}{:=}(x_{53},0)^\star$
T_4	$X_{14}{:=}\left(\binom{x_{15}}{0},0\right)$ $Y_{14}{:=}\left(\binom{0}{x_{15}},0\right)$			$\left(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),1\right)$	$X_{54}{:=}\left(\binom{0}{1},0\right)$ $Y_{54}{:=}\left(\binom{1}{0},0\right)$
T_5	$X_{15}{:=}(x_{15},0)$ $Y_{15}{:=}(x_{14}x_{45},0)$ $Z_{15}{:=}(x_{14}y_{45},0)$		$X_{35}{:=}(x_{34}x_{45},0)$ $Y_{35}{:=}(x_{34}y_{45},0)^\star$		$(1,0)$

Table 4: End T for dualising on node 4. Massive fields marked $\star$.



(a) Dual quiver with massive fields. (b) Dual quiver.

Figure 6: Quivers from Seiberg dualising on node 5.

which can be seen to be zero in the derived category from the following diagrams respectively

$$0 \longrightarrow \underline{P_5 \oplus P_5} \xrightarrow{\binom{x_{45}}{-y_{45}}} P_4 \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow \underline{P_5 \oplus P_5} \xrightarrow{\binom{x_{45}}{-y_{45}}} P_4 \longrightarrow 0 \tag{3.28}$$

$$\begin{array}{ccccccccc} & & & & \underline{P_5 \oplus P_5} & & & & \\ & & & \swarrow^{\binom{x_{14}x_{45}}{-x_{14}y_{45}}} & & \searrow^{x_{14}} & & & \\ 0 & \longrightarrow & & & \underline{P_1} & & & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccccccc} & & & & \underline{P_5 \oplus P_5} & & & & \\ & & & \swarrow^{\binom{x_{34}x_{45}}{-x_{34}y_{45}}} & & \searrow^{x_{34}} & & & \\ 0 & \longrightarrow & & & \underline{P_3} & & & \longrightarrow & 0 \end{array}$$

3.5 Dualising on the fifth node

Finally, we dualise the quiver Q in figure 1 by performing Seiberg duality on the fifth node. Again, we apply the rules (2.1) to derive the adjacency matrix of the dual quiver as

$$\widetilde{C} = \begin{pmatrix} 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & -4 & -1 & 0 & 2 \\ 0 & 2 & 1 & 0 & 0 \end{pmatrix}. \tag{3.29}$$

The corresponding dual quiver is shown in figure 6b. The one with massive fields is shown in figure 6a. The path algebra of the dual quiver figure 6b is obtained as the endomorphism algebra of the tilting complex $T = \oplus_{i=1}^{5} T_i$ in the derived category $D(Q)$, with direct summand

$$T_1 : \quad 0 \longrightarrow \underline{P_1} \longrightarrow 0$$

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	T_1	T_2	T_3	T_4	T_5
T_1	$(1,0)$	$X_{21}{:=}(x_{21},0)^\star$	$X_{31}{:=}(x_{31},0)^\star$		$X_{51}{:=}\left(\begin{pmatrix}0\\ x_{21}\\ -x_{31}\end{pmatrix},0\right)$
T_2	$X_{12}{:=}(x_{15}x_{52},0)$ $Y_{12}{:=}(x_{15}y_{52},0)^\star$	$(1,0)$		$X_{42}{:=}(x_{45}x_{52},0)$ $Y_{42}{:=}(x_{45}y_{52},0)$ $Z_{42}{:=}(y_{45}x_{52},0)$ $W_{42}{:=}(y_{45}y_{52},0)$	
T_3	$X_{13}{:=}(x_{15}x_{53},0)^\star$	$X_{23}{:=}(x_{23},0)$	$(1,0)$	$X_{43}{:=}(x_{45}x_{53},0)$ $Y_{43}{:=}(y_{45}x_{53},0)^\star$	
T_4	$X_{14}{:=}(x_{14},0)$		$X_{34}{:=}(x_{34},0)^\star$	$(1,0)$	$X_{54}{:=}\left(\begin{pmatrix}x_{21}x_{14}\\ 0\\ -x_{34}\end{pmatrix},0\right)$ $Y_{54}{:=}\left(\begin{pmatrix}x_{23}x_{31}x_{14}\\ -x_{23}x_{34}\\ 0\end{pmatrix},0\right)$
T_5		$X_{25}{:=}\left(\begin{pmatrix}1\\0\\0\end{pmatrix},0\right)$ $Y_{25}{:=}\left(\begin{pmatrix}0\\1\\0\end{pmatrix},0\right)$ $Z_{25}{:=}\left(\begin{pmatrix}0\\0\\x_{23}\end{pmatrix},0\right)$	$X_{35}{:=}\left(\begin{pmatrix}0\\0\\1\end{pmatrix},0\right)$		$\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},1\right)$

Table 5: End T for dualising on node 5. Massive fields marked $\star$.

$$
\begin{aligned}
T_2:&\quad 0\longrightarrow\underline{P_2}\longrightarrow 0\\
T_3:&\quad 0\longrightarrow\underline{P_3}\longrightarrow 0\\
T_4:&\quad 0\longrightarrow\underline{P_4}\longrightarrow 0\\
T_5:&\quad 0\longrightarrow\underline{P_2\oplus P_2\oplus P_3}\xrightarrow{\begin{pmatrix}x_{52}\\ y_{52}\\ x_{53}\end{pmatrix}}P_5\longrightarrow 0\,.
\end{aligned}
\tag{3.30}
$$

The homomorphisms in End$T$ are shown in table 5. From the table we note that $Z_{25} = X_{23}X_{35}$ and is thus dropped. Moreover, from the superpotential $\mathcal{W}$, the terms $y_{45}x_{53}x_{34} = Y_{43}X_{34}$, $x_{15}x_{53}x_{21} = X_{13}X_{31}$ and $x_{15}y_{52}x_{21} = Y_{12}X_{21}$, thus giving masses to the dual quarks.

The calculation of the dual superpotential for this case presents some interesting subtleties. The rules mentioned in page 8 can not be used to fix the dual superpotential. In order to evaluate the dual superpotential systematically, we first write down all possible terms in the superpotential and then retain the ones which will be compatible with the relations between the fields in table 5. Indeed, we use only a few relations to determine the superpotential completely. The other relations can then be checked to be satisfied, thus ensuring the consistency of the procedure.

Let us discuss some details of this calculation in brief. The dual quarks in the gauge theory, as can be read off from table 5, are X_{25}, Y_{25}, X_{35}, X_{51}, X_{54} and Y_{54} while the mesons are X_{12}, Y_{12}, X_{13}, X_{42}, Y_{42}, Z_{42}, W_{42}, X_{43} and Y_{43}. First, we express the superpotential $\mathcal{W}$ in terms of the dual fields from the table, as usual. It assumes the following form

$$
\mathcal{W}' = Y_{42}X_{23}X_{34} + X_{14}Z_{42}X_{21} + X_{31}X_{13} - Y_{43}X_{34} - X_{42}X_{23}X_{31}X_{14} - Y_{12}X_{21}\,. \tag{3.31}
$$

Clearly, X_{34}, Y_{43}, X_{31} and X_{13} have superpotential mass. Before writing down the mesonic part let us note that the massive fields X_{34}, X_{21} and X_{31} are not in the above list of dual quarks and mesons. Hence they will not appear in the mesonic part $\mathcal{W}_m$ of the superpotential. The variations of the dual superpotential with respect to these fields,

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therefore, have no other possible contribution than those arising from the variations of $\mathcal{W}'$. The variations of $\mathcal{W}'$ with respect to X_{34}, X_{21} and X_{31} yield

$$Y_{43} = Y_{42}X_{23}\,, \qquad Y_{12} = X_{14}Z_{42}\,, \qquad X_{13} = X_{14}X_{42}X_{23}\,, \tag{3.32}$$

respectively. Substituting these in the expression of $\mathcal{W}'$ we find that the superpotential $\mathcal{W}'$ vanishes. This means that the dual superpotential is expressed solely in terms of the perturbations of the original superpotential. This feature is unique to the case in hand among the examples studied in this note.

The dual superpotential thus consists only of the mesonic part. There are twenty five terms in the mesonic part, consisting of combinations of the dual quarks and mesons listed above. In writing the mesonic, alias dual, superpotential we have to decide which of these twenty five terms will be retained as part of $\mathcal{W}_m$ and also determine their signs. It can be checked that the rules mentioned in page 8 fail in this case to determine all the terms. In order to find out this part of the superpotential we first write down all the twenty five terms with both the possible signs for each. The resulting potential takes the following form.

$$\begin{aligned}
\widehat{W} = {} & \pm X_{54}X_{43}X_{35} \pm Y_{54}X_{43}X_{35} \pm X_{54}Y_{43}X_{35} \pm Y_{54}Y_{43}X_{35} \pm \\
& \pm \underline{X_{51}X_{12}X_{25}} \pm X_{51}X_{12}Y_{25} \pm X_{51}Y_{12}X_{25} \pm \underline{X_{51}Y_{12}Y_{25}} \pm \underline{X_{51}X_{13}X_{35}} \pm \\
& \pm X_{54}X_{42}X_{25} \pm X_{54}X_{42}Y_{25} \pm Y_{54}X_{42}X_{25} \pm Y_{54}X_{42}Y_{25} \pm \\
& \pm X_{54}Y_{42}X_{25} \pm X_{54}Y_{42}Y_{25} \pm Y_{54}Y_{42}X_{25} \pm Y_{54}Y_{42}Y_{25} \pm \\
& \pm X_{54}Z_{42}X_{25} \pm X_{54}Z_{42}Y_{25} \pm Y_{54}Z_{42}X_{25} \pm Y_{54}Z_{42}Y_{25} \pm \\
& \pm X_{54}W_{42}X_{25} \pm X_{54}W_{42}Y_{25} \pm Y_{54}W_{42}X_{25} \pm Y_{54}W_{42}Y_{25}\,.
\end{aligned} \tag{3.33}$$

We then consider variations of this new potential with respect to some of the fields. From table 5 we find out which of the terms may be retained in order for these variations to vanish. Let us illustrate this procedure with two examples.

There are only two terms in $\mathcal{W}'$ which contain X_{43}, namely the first two in (3.33). Variation of $\widehat{\mathcal{W}}$ with respect to X_{43} should yield the F-flatness conditions to be satisfied by the fields in table 5. From the table we find that $X_{35}X_{54} = -x_{34}$, while $X_{35}Y_{54} = 0$. Hence only the second one can be retained. The sign is not yet fixed, however. Next, let us consider the terms containing X_{51} in $\widehat{W}$. There are five such terms, written in the second line in (3.33). Varying with respect to X_{51} we find that the variation can be made equal to

$$\psi := x_{15} \cdot \begin{pmatrix} x_{52} \\ y_{52} \\ x_{53} \end{pmatrix} \tag{3.34}$$

if we retain only the three terms underlined with same sign for all the three. Here we used the F-flatness conditions of the superpotential $\mathcal{W}$. This morphism is zero in the homotopy category of the complexes T_i, as can be seen from the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{P_2 \oplus P_2 \oplus P_3} & \xrightarrow{\begin{pmatrix} x_{52} \\ y_{52} \\ x_{53} \end{pmatrix}} & P_5 & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \psi} & \swarrow{\scriptstyle x_{15}} & & & \\
0 & \longrightarrow & \underline{P_1} & \longrightarrow & 0 & &
\end{array} \tag{3.35}$$

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Repeating this exercise considering variations of $\widehat{W}$ with respect to X_{54} and Y_{54}, we can single out the terms which appear in the dual superpotential, not completely fixed yet though. The potential $\widehat{W}$ after dropping out the extra terms from (3.33) takes the form

$$\begin{aligned}\widehat{W} = &\alpha(X_{51}X_{12}X_{25} + X_{51}X_{14}X_{42}X_{23}X_{35} + X_{51}X_{14}Z_{42}Y_{25}) \\ &\beta(X_{54}Z_{42}X_{25} + X_{54}W_{42}Y_{25} + X_{54}Y_{42}X_{23}X_{35}) \\ &\gamma(Y_{54}X_{42}X_{25} + Y_{54}Y_{42}Y_{25} + Y_{54}X_{43}X_{35})\,,\end{aligned} \tag{3.36}$$

where we have used the relations (3.32) to substitute the expressions for the fields determined in terms of others. Here α, β, γ are signs (± 1) to be determined. To this end, we consider the variation of $\widehat{W}$ with respect to X_{35}. Using the expressions of the dual fields from table 5 we write the variation in terms of the original fields to obtain

$$\frac{\partial \widehat{W}}{\partial X_{35}} = \begin{pmatrix} \beta \cdot x_{21}x_{14}x_{45}y_{52}x_{23} + \gamma \cdot x_{23}x_{31}x_{14}x_{45}x_{53} \\ \alpha \cdot x_{21}x_{14}x_{45}x_{52}x_{23} - \gamma \cdot x_{23}x_{34}x_{45}x_{53} \\ -\alpha \cdot x_{31}x_{14}x_{45}x_{52}x_{23} - \beta \cdot x_{34}x_{45}y_{52}x_{23} \end{pmatrix}. \tag{3.37}$$

Now, using the F-flatness conditions ensuing from the variations of $\mathcal{W}$, we can set this variation to zero by choosing $\alpha = 1$, $\beta = -1$ and $\gamma = 1$. This fixes all the signs in $\widehat{W}$, which yields the mesonic superpotential $\mathcal{W}_m$, which in turn is the dual superpotential $\widetilde{\mathcal{W}} = 0$. That is, finally we have obtained the dual superpotential

$$\begin{aligned}\widetilde{\mathcal{W}} = &X_{51}X_{12}X_{25} + X_{51}X_{14}X_{42}X_{23}X_{35} + X_{51}X_{14}Z_{42}Y_{25} - X_{54}Z_{42}X_{25} - X_{54}W_{42}Y_{25} - \\ &-X_{54}Y_{42}X_{23}X_{35} + +Y_{54}X_{42}X_{25} + Y_{54}Y_{42}Y_{25} + Y_{54}X_{43}X_{35}\,.\end{aligned} \tag{3.38}$$

Let us point out that we have used variations of the potential $\widehat{W}$ with respect to only five out of the fourteen dual fields to fix the dual superpotential $\widetilde{\mathcal{W}}$. It can now be verified that the variations of the dual superpotential with respect to all the dual fields vanish. In other words, the relations arising from the F-flatness conditions of the dual theory are satisfied by the morphisms in table 5.

4. The general picture

From the combinatorics of the examples discussed above and earlier ones [17, 19] there emerges a general scheme to obtain Seiberg dual quiver gauge theories with superpotentials. Let us conclude this note by discussing this scheme. The idea is to start with a quiver Q along with a superpotential $\mathcal{W}$ of the associated $\mathsf{N} = 1$ gauge theory. We choose one node of the quiver as the dualising node, denoted i_0. We then write down a tilting complex T in the bounded derived category $D(Q)$ of the path algebra of Q using the incoming arrows on i_0. The tilting complex T has the form $\oplus_{i=1}^{N} T_i$, where N equals the number of nodes in Q, where each direct summand T_i consists of the projective P_i from the path algebra Q-mod in the zeroth degree, if $i \neq i_0$. The summand T_{i_0} is a two-term complex, which is of the form

$$T_{i_0}: \quad 0 \longrightarrow \underline{\oplus_{j \in I_{\rm in}} P_j} \xrightarrow{(\mathfrak{p}_{i_0 j})} P_{i_0} \longrightarrow 0\,, \tag{4.1}$$

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where $j \in I_{\rm in}$ and $\mathfrak{p}_{i_0 j}$ denote the morphisms in Q-mod corresponding to the arrows terminating on i_0 as well as the associated chiral fields. By interpreting the endomorphism algebra of T as the path algebra of another quiver, we obtain the dual quiver $\widetilde{Q}$. We now write down the dual superpotential starting from the original one $\mathcal{W}$.

To this end let us define $\mathfrak{q}_{k i_0}$, with $k \in I_{\rm out}$ to denote the morphisms corresponding to the arrows emanating from i_0 and the associated fields. The relevant part of the superpotential of the gauge theory associated to the quiver Q is

$$\mathcal{W} = \mathfrak{p} \cdot \mathfrak{M} \cdot \mathfrak{q}\,, \tag{4.2}$$

where, again we refrain from mentioning the trace over gauge indices explicitly. Here $\mathfrak{M}_{jk}$ is a matrix, in general, with $j \in I_{\rm out}$ and $k \in I_{\rm in}$ corresponding to (possibly composite) morphism in Q-mod from the k-th to the j-th node. The F-flatness conditions of $\mathcal{W}$ imposes $\mathfrak{p} \cdot \mathfrak{M} = 0 = \mathfrak{M} \cdot \mathfrak{q}$.

Similarly, we define dual fields corresponding to the morphisms in $\widetilde{Q}$-mod, which appear in the dual superpotential. The dual quarks are denoted by the morphisms $\widetilde{\mathfrak{p}}$ and $\widetilde{\mathfrak{q}}$, in the same convention as in the last paragraph and each of these corresponds to a moiety of $\mathfrak{M}$ as

$$\mathfrak{M} = \widetilde{\mathfrak{q}} \cdot \widetilde{\mathfrak{p}}\,. \tag{4.3}$$

In order to find the dual fields, we need to solve (4.3). The solution is not necessarily unique; the dual fields introduced earlier correspond to some consistent choice of solutions for this equation. In fact, this equation does not make sense in the special cases in which $\mathfrak{M}$ is a single field, as occurs in considering the McKay quiver corresponding to the orbifold $\mathbb{C}^3/\mathbb{Z}_3$. Generally, the dual superpotential can be written in terms of these dual fields by introducing the mesons

$$\widetilde{\mathfrak{M}} := \mathfrak{q} \cdot \mathfrak{p}\,. \tag{4.4}$$

The expressions (4.3) and (4.4) clearly bring out the duality between the fields we are discussing. We write the dual superpotential as a sum of the original superpotential, written in terms of the dual fields and the mesonic part, that is the deformation of the original superpotential, as

$$\widetilde{\mathcal{W}} = \mathfrak{M} \cdot \widetilde{\mathfrak{M}} - \widetilde{\mathfrak{p}} \cdot \widetilde{\mathfrak{M}} \cdot \widetilde{\mathfrak{q}}\,, \tag{4.5}$$

where in the first term, in re-expressing the original superpotential $\mathcal{W}$ we used cyclic properties of the trace over gauge indices. Moreover, $\mathfrak{M}$ must be expressed in terms of the dual fields in this expression.

Finally, let us verify that the dual fields thus defined satisfy relations ensuing from the F-flatness conditions of the dual superpotential (4.5). The F-flatness conditions obtained from the variations of $\widetilde{\mathcal{W}}$ with respect to the mesons $\widetilde{\mathfrak{M}}$ are of the form $\mathfrak{M} - \widetilde{\mathfrak{q}} \cdot \widetilde{\mathfrak{p}} = 0$, which is satisfied by virtue of (4.3), by construction. The ones ensuing from the variations with respect to the dual quarks $\widetilde{\mathfrak{p}}$ or $\widetilde{\mathfrak{q}}$ involve $\widetilde{\mathfrak{M}}$. These can be shown to be zero in the

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homotopy category of the complexes, and hence in the derived category $D(Q')$, since $\widetilde{\mathfrak{M}}$ is a morphism homotopic to zero as can be shown by considering the diagram with $k \in I_{\rm in}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\oplus_{j\in I_{\rm in}} P_j} & \stackrel{\mathfrak{p}}{\longrightarrow} & P_{i_0} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \widetilde{\mathfrak{M}}} & \swarrow{\scriptstyle \mathfrak{q}} & & & \\ 0 & \longrightarrow & P_k & \longrightarrow & 0 & & \end{array} \tag{4.6}$$

generally, by using (4.4). This indeed is what happened in the examples of the last section. This furnishes a general method to obtain Seiberg dual quiver gauge theories, exemplified by the ones in the last section. Let us point out that this method does not depend on the underlying geometry of the vacuum moduli space of the gauge theory.

5. Summary

In this note we have obtained Seiberg dual quiver gauge theories with $\mathsf{N} = 1$ supersymmetry as theories associated to quivers whose path algebras are derived equivalent. The vacuum moduli space of these gauge theories are the same, namely, the complex cone over the second del Pezzo surface. We have established the derived equivalence of the path algebras by explicitly constructing tilting complexes in the derived category of the path algebras. The endomorphisms of the tilting complex furnish the dual quarks and mesons of the dual theory and satisfy the F-flatness conditions derived from the corresponding dual superpotentials. In one of the examples we found that it even turns out to be easier to obtain the dual superpotential from the endomorphism algebra of the tilting complex. This furnishes a non-trivial explicit verification of the conjecture that Seiberg duality is a derived equivalence of physical gauge theories with superpotentials [17]. We constructed the tilting complexes explicitly for five different cases, by dualising on each node in turn. As expected, one of the new quivers thus obtained matches with the original one up to permutation of nodes. One other matches with the quiver obtained by toric duality.

We then verified in each case that the endomorphism algebra of the resulting tilting complex yields a Seiberg dual theory. This, in fact, turns out to be a general pattern in writing the tilting complexes in similar cases. On the basis of the examples in this note as well as the ones studied earlier, we presented a general scheme for obtaining Seiberg dual theories with superpotentials. In particular, we showed how to construct the dual quiver from a tilting complex in the path algebra of the original quiver one starts with and presented a method to write the dual superpotential. We showed that the F-flatness conditions ensuing from the dual superpotential are satisfied by the morphisms in the endomorphism algebra of the dual quiver in general.

We have, by this procedure, obtained Seiberg dual theories, which are not toric. This exemplifies the fact that derived equivalence is more general than toric duality. The rules (2.1) obtained earlier, however, apply to all these cases and in fact, the morphisms in $\mathrm{End}\, T$ are precisely counted by these rules. This enables one to explore the complete moduli space of $\mathsf{N} = 1$ gauge theories associated with representations of a quiver. In this note we obtained at different phases of the moduli space of gauge theories associated with

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the second del Pezzo surface which were unreachable by toric duality alone. These phases are connected by renormalisation group flows.

While not unexpected on general grounds, it would be interesting to explicitly check if the general pattern of obtaining Seiberg dual theories mentioned above work also for other non-trivial quiver theories. It would be nice to have a proof of the fact that, although the general construction does not depend directly on the underlying geometry of the vacuum moduli space of the gauge theory, the construction indeed produce gauge theories with the same moduli space. As this approach is algebraic not depending on the detailed geometry of the underlying space, it would be interesting to see if this sheds some light on Seiberg duality in the presence of fluxes, for example, theories with discrete torsion. We hope to return to some of these issues in future.

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References

[1] N. Seiberg, *Electric-magnetic duality in supersymmetric nonabelian gauge theories*, *Nucl. Phys.* **B 435** (1995) 129 [hep-th/9411149].

[2] J.H. Brodie and M.J. Strassler, *Patterns of duality in $N = 1$ SUSY gauge theories or: seating preferences of theater-going non-abelian dualities*, *Nucl. Phys.* **B 524** (1998) 224 [hep-th/9611197].

[3] A. Giveon and D. Kutasov, *Brane dynamics and gauge theory*, *Rev. Mod. Phys.* **71** (1999) 983 [hep-th/9802067].

[4] K.A. Intriligator, R.G. Leigh and M.J. Strassler, *New examples of duality in chiral and nonchiral supersymmetric gauge theories*, *Nucl. Phys.* **B 456** (1995) 567 [hep-th/9506148].

[5] A. Hanany and E. Witten, *Type-IIB superstrings, BPS monopoles and three-dimensional gauge dynamics*, *Nucl. Phys.* **B 492** (1997) 152 [hep-th/9611230].

[6] D.-E. Diaconescu, *D-branes, monopoles and Nahm equations*, *Nucl. Phys.* **B 503** (1997) 220 [hep-th/9608163].

[7] S. Elitzur, A. Giveon, D. Kutasov, E. Rabinovici and A. Schwimmer, *Brane dynamics and $N = 1$ supersymmetric gauge theory*, *Nucl. Phys.* **B 505** (1997) 202 [hep-th/9704104].

[8] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167; For an earlier work on array of gauge theories see: M.B. Halpern and W. Siegel, *Electromagnetism as a strong interaction*, *Phys. Rev.* **D 11** (1975) 2969.

[9] B. Feng, A. Hanany and Y.-H. He, *D-brane gauge theories from toric singularities and toric duality*, *Nucl. Phys.* **B 595** (2001) 165 [hep-th/0003085].

[10] C.E. Beasley and M.R. Plesser, *Toric duality is Seiberg duality*, *J. High Energy Phys.* **12** (2001) 001 [hep-th/0109053].

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[11] B. Feng, A. Hanany and Y.-H. He, *Phase structure of D-brane gauge theories and toric duality*, *J. High Energy Phys.* **08** (2001) 040 [hep-th/0104259].

[12] B. Feng, A. Hanany, Y.-H. He and A.M. Uranga, *Toric duality as Seiberg duality and brane diamonds*, *J. High Energy Phys.* **12** (2001) 035 [hep-th/0109063].

[13] B. Feng, S. Franco, A. Hanany and Y.-H. He, *Unhiggsing the del Pezzo*, *J. High Energy Phys.* **08** (2003) 058 [hep-th/0209228].

[14] S. Franco and A. Hanany, *Toric duality, Seiberg duality and Picard-Lefschetz transformations*, *Fortschr. Phys.* **51** (2003) 738 [hep-th/0212299].

[15] T. Sarkar, *D-brane gauge theories from toric singularities of the form* $\mathbb{C}^3/\Gamma$ *and* $\mathbb{C}^4/\Gamma$, *Nucl. Phys.* **B 595** (2001) 201 [hep-th/0005166].

[16] D.R. Morrison and M.R. Plesser, *Non-spherical horizons, I*, *Adv. Theor. Math. Phys.* **3** (1999) 1 [hep-th/9810201].

[17] D. Berenstein and M.R. Douglas, *Seiberg duality for quiver gauge theories*, hep-th/0207027.

[18] J Rickard, *Morita theory for derived categories*, *J. Lond. Math. Soc.(2)* **39(3)** (1989) 436.

[19] V. Braun, *On Berenstein-Douglas-Seiberg duality*, *J. High Energy Phys.* **01** (2003) 082 [hep-th/0211173].

[20] S. Franco, A. Hanany, Y.-H. He and P. Kazakopoulos, *Duality walls, duality trees and fractional branes*, hep-th/0306092.

[21] B. Fiol, *Duality cascades and duality walls*, *J. High Energy Phys.* **07** (2002) 058 [hep-th/0205155].

[22] A. Hanany and J. Walcher, *On duality walls in string theory*, *J. High Energy Phys.* **06** (2003) 055 [hep-th/0301231].

[23] M.R. Douglas, *D-branes, categories and* $N = 1$ *supersymmetry*, *J. Math. Phys.* **42** (2001) 2818 [hep-th/0011017].

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